ARKIV FÖR FYSIK Band 18 nr 10

Communicated 24 February 1960 by E. Rudberg and F. Odqvist

The propagation of a brittle crack

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With 16 figures in the text

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SUMMARY

The propagation characteristics of a crack in a brittle, linear elastic material have been investigated. In order to obtain a mathematical description, the material is supposed to be homogeneous and isotropic both as regards stress-strain relations and fracturing characteristics. It is à priori assumed that a two-dimensional crack can be propagated and that this crack propagates in one plane, only. Further, the surface energy of the material is supposed to be zero. Then the crack-tip velocity will be constant (cf. Mott, 1948), and the problem of the crack propagation can be investigated by a study of displacements and stresses in a semi-infinite solid, where the plane surface is subjected

1. to a constant pressure, acting on an infinite strip, the width of which is symmetrically increasing from zero with a constant velocity, and
2. to a pressure outside this strip such that the normal displacement of the surface outside this strip is zero.

This mixed boundary value problem has been treated. The stresses in the solid and the normal displacement of the surface have been solved exactly.

The result shows that the displacement of the surface is elliptic, just as in the corresponding static case. By studying the stresses in the vicinity of the edge of the strip and using some simple fracture criteria it can be shown that the assumption à priori regarding the existence of a two-dimensional crack propagating in one plane is justified. Further it can be concluded that the crack-tip velocity equals the velocity of surface (Rayleigh) waves.

In a real material, the existence of a surface energy does not seem to have much influence on the crack propagation except for the very start (Mott, 1948). The inhomogeneity and the anisotropy of the fracturing characteristics will cause deviations from a plane propagation of the crack. This fact will reduce the mean velocity of the crack in the direction of propagation. The effects of non-brittleness or deviations from Hooke's law are not discussed.

Introduction

By using dimensional analysis Mott (1948) was able to show that the maximum propagation velocity of a crack is independent of the surface energy. If we consider a two-dimensional crack of length 2 a, propagating symmetrically with constant crack-tip velocity

$$\frac{da}{dt}=v_c$$

then the surface energy is proportional to a, whereas the release of stress energy due to the propagation of the crack and the kinetic energy resulting from this propagation are both proportional to a^2. Thus the energy balance is simply a balance between the released stress energy and the kinetic energy when $a \to \infty$. Roberts and Wells (1954) assumed that the stress state in the material surrounding the crack is approximately equal to the corresponding stress state in the static case. Using Mott's criterion for the energy balance they arrived at the conclusion that the maximum crack velocity in a brittle material is about 38 per cent of the sound velocity. The approximation used by Roberts and Wells does not, however, seem to be permissible.

A two-dimensional crack will be considered in the following. The material is assumed to be homogeneous and isotropic both as regards fracturing characteristics and stress-strain relations. It is supposed to obey Hooke's law and to be perfectly brittle. Further the surface energy is assumed to be zero. This assumption simplifies the study, because the crack can be supposed to nucleate from an infinitesimally small micro-crack and to propagate with maximum velocity from the start. On the other hand, as the maximum velocity obtained is independent of the surface energy this assumption does not violate our possibility to find the maximum crack velocity in a brittle material.

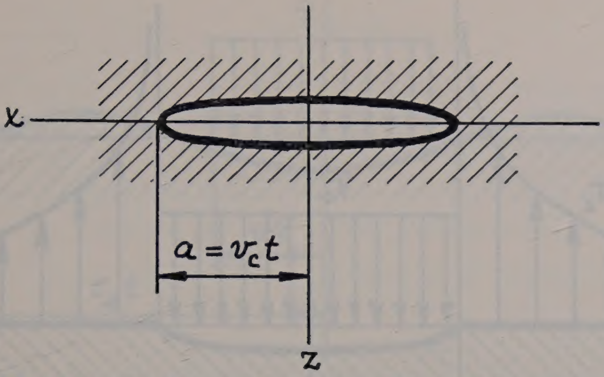


Fig. 1. Coordinate system and position of crack at time t. The negative y-axis points towards the reader.

Consider an infinite medium with the characteristics specified above. Cartesian coordinates x, y, z are introduced (see Fig. 1). The crack is supposed to nucleate from an infinitesimally small micro-crack situated along the y-axis from $y = -\infty$ to $y = +\infty$ and to propagate symmetrically in positive and negative x-directions with the constant crack-tip velocity v_c when an applied tensile stress reaches the nominal value

$$\sigma_z = q_0.$$

Here we assume a state of plane strain, so that nominally

$$\sigma_x = \sigma_y = \frac{\nu}{1-\nu} q_0 \quad (\nu = \text{Poisson's ratio})$$

$$\tau_{xy} = \tau_{yz} = \tau_{zx} = 0.$$

We intend to find the crack-tip velocity v_c, the shape of the crack, and the stresses in the solid.

Method of treating the problem

When treating the problem mathematically it is convenient to superpose a state of stress

$$\sigma_z = -q_0$$

$$\sigma_x = \sigma_y = -\frac{\nu}{1-\nu} q_0$$

$$\tau_{xy} = \tau_{yz} = \tau_{zx} = 0$$

defining a state of plane strain. Inside the crack, where the z-component, only, is of importance, we may suppose that the superposition is achieved by the action of a uniform liquid pressure q_0.

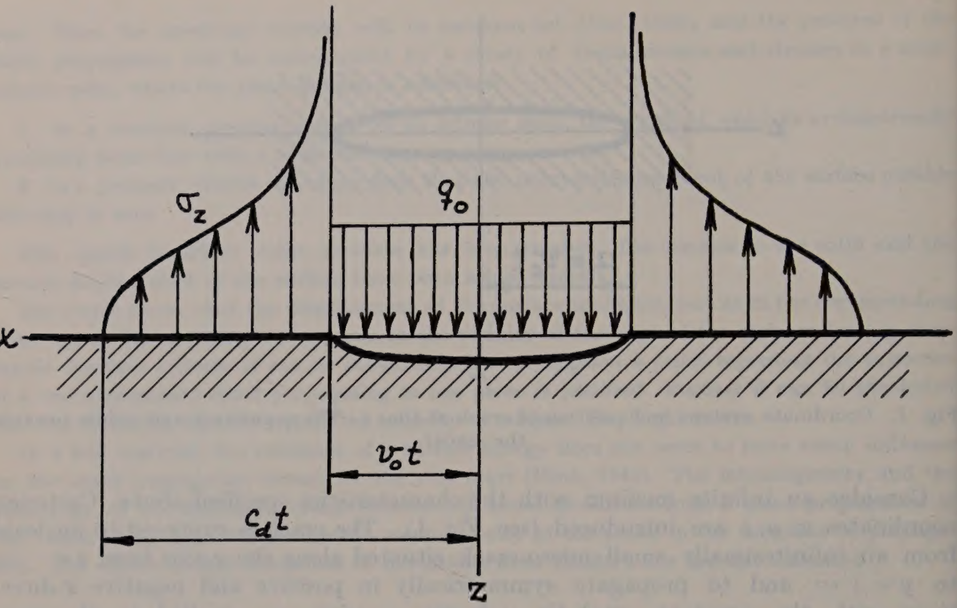


Fig. 2. Load and surface displacement at time t in Problem A.

After this state of stress has been superposed, the stresses at infinity are all zero. A cut is made along the plane $z=0$ and the semi-infinite solid $z \geqslant 0$ is considered. Thus the following problem is created (see Fig. 2):

Problem A.—A pressure q_0 is acting on the infinite strip $|x| < v_0 t$ of the surface $z=0$ of a semi-infinite solid $z \geqslant 0$. The vertical displacement u_z at the surface $z=0$ is zero for $|x| \geqslant v_0 t$. Find the normal stress σ_z at the surface $z=0$ for $|x| > v_0 t$ and the displacement u_z at the surface $z=0$ for $|x| < v_0 t$.

After solving this problem, the original problem can be treated after investigation of the stresses in the vicinity of the crack-tip, using a suitable fracture criterion.

In order to solve Problem A, the following problem is first solved (see Fig. 3):

Problem B.—A pressure q is acting on the infinite strip $|x| < vt$ of the surface $z=0$ of a semi-infinite solid $z \geqslant 0$, whereas no pressure acts on the surface $z=0$ for $|x| > vt$. Find the displacement u_z at the surface $z=0$.

After solving this problem, a superposition of different pressures q with different propagation velocities v can be made in such a way that the resulting pressure is q_0 for $|x| < v_0 t$ and the resulting displacement u_z at the surface $z=0$ is zero for $|x| \geqslant v_0 t$. This leads to an integral equation for the normal stress σ_z on the surface $z=0$ for $|x| > v_0 t$. Solving this integral equation, the first part of Problem A is solved. This part of the problem will be called *Problem C.*

After Problem C is solved, the second part of Problem A is solved, viz. to find the displacement u_z at the surface $z=0$ for $|x| < v_0 t$. This part of the problem will be called *Problem D.*

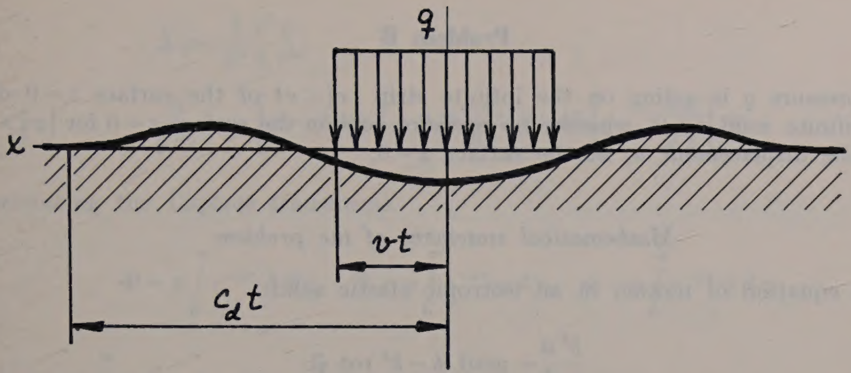


Fig. 3. Load and surface displacement at time t in Problem B.

In fact, instead of working out the displacement u_z in Problem B, the acceleration $\partial^2 u_z/\partial t^2$ will be calculated and used for the superposition in Problem C, because this simplifies the calculations. It is known that if $\partial^2 u_z/\partial t^2 = 0$ for $|x| > v_0 t$, then $u_z = 0$ for $|x| > v_0 t$ because $u_z = 0$ for $|x| > c_d t$ (c_d = propagation velocity of irrotational waves).

Symbols

x, y, z: Cartesian coordinates. The semi-infinite solid occupies the space $z \geqslant 0$.

$\hat{x}$, $\hat{y}$, $\hat{z}$: Unit vectors in positive x-, y- and z-directions, respectively.

$\bar{u} = |u_x, u_y, u_z|$: Displacement vector. Conditions of plane strain are prevailing, so that $u_y = 0$ and u_x and u_z are functions of x and z, only.

$\sigma_x, \sigma_y, \sigma_z, \tau_{xy}, \tau_{yz}, \tau_{zx}$: Components of stress tensor. As the solid obeys Hooke's law, $\tau_{xy} = \tau_{yz} = 0$.

c_d: Propagation velocity of irrotational waves.

c_r: Propagation velocity of equivoluminal waves.

c_s: Propagation velocity of surface (Rayleigh) waves.

$$k = \frac{c_r}{c_d} < 1$$

$$k_s = \frac{c_s}{c_d} < k$$

t: Time

$\tau = c_d\, t$

p: Laplace transform variable with respect to τ.

$$U(t) = \left\{ \begin{array}{ll} 0 & \text{for } t < 0 \\ {}^1/_2 & \text{for } t = 0 \\ 1 & \text{for } t > 0 \end{array} \right\}: \text{ Unit step function}$$

$\delta(t) = \dfrac{d}{dt} U(t)$: Dirac's delta function.

$\phi = \operatorname{div} \bar{u}$

$\bar{\psi} = \operatorname{rot} \bar{u}$

ϱ: Density of the solid.

Other symbols are defined in the text.

Problem B

A pressure q is acting on the infinite strip $|x| < vt$ of the surface $z = 0$ of a semi-infinite solid $z \geqslant 0$, whereas no pressure acts on the surface $z = 0$ for $|x| > vt$. Find the displacement u_z at the surface $z = 0$.

Mathematical statement of the problem

The equation of motion in an isotropic elastic solid,

$$\frac{\partial^2 \bar{u}}{\partial \tau^2} = \operatorname{grad} \phi - k^2 \operatorname{rot} \bar{\psi} \tag{1}$$

can be split up into the equations

$$\operatorname{div} \operatorname{grad} \phi = \frac{\partial^2 \phi}{\partial \tau^2} \tag{2}$$

$$- \operatorname{rot} \operatorname{rot} \bar{\psi} = \frac{1}{k^2} \cdot \frac{\partial^2 \bar{\psi}}{\partial \tau^2} \tag{3}$$

On the boundary $z = 0$ the following relation holds

$$\frac{\bar{\sigma}}{\varrho c_d^2} = -(1 - 2k^2)\, \phi\, \hat{z} + k^2\, \bar{\psi} \times \hat{z} - 2k^2 \frac{\partial \bar{u}}{\partial z}, \tag{4}$$

where $\bar{\sigma}$ = stress vector referring to the surface $z = 0$, i.e.

$$\bar{\sigma} = \begin{cases} q\,\hat{z} & \text{for } |x| < \beta\tau \\ 0 & \text{for } |x| > \beta\tau, \end{cases}$$

where
$$\beta = \frac{v}{c_d}.$$

Now, because $\bar{u}$ is a function of x and z, only:

$$\phi = \phi(x, z)$$

$$\bar{\psi} = \psi(x, z) \cdot \hat{y}.$$

Thus Eqs. (1), (2), (3), (4) turn into

$$\frac{\partial^2 \bar{u}}{\partial \tau^2} = \operatorname{grad} \phi - k^2 \left(\frac{\partial \psi}{\partial z} \hat{x} - \frac{\partial \psi}{\partial x} \hat{z} \right)$$

$$\Delta \phi = \frac{\partial^2 \phi}{\partial \tau^2}$$

$$\Delta\psi=\frac{1}{k^2}\frac{\partial^2\psi}{\partial\tau^2}$$

$$\frac{\bar\sigma}{\varrho\,c_d^2}=-\left[(1-2k^2)\,\phi+2k^2\frac{\partial u_z}{\partial z}\right]\hat z+k^2\left[\psi-2\frac{\partial u_x}{\partial z}\right]\hat x.$$

Introducing the Laplace transforms

$$\Phi=p\int_0^\infty e^{-p\tau}\phi\,d\tau,\quad \Psi=p\int_0^\infty e^{-p\tau}\psi\,d\tau,\quad \bar w=p\int_0^\infty e^{-p\tau}\bar u\,d\tau$$

and $S_z=p\int_0^\infty e^{-p\tau}\sigma_z\,d\tau$ one obtains the equations

$$(S_z)_{z=0}=-q\,e^{-p|x|/\beta} \tag{5}$$

$$p^2\,\bar w=\left(\frac{\partial\Phi}{\partial x}+k^2\frac{\partial\Psi}{\partial z}\right)\hat x+\left(\frac{\partial\Phi}{\partial z}-k^2\frac{\partial\Psi}{\partial x}\right)\hat z \tag{6}$$

$$\Delta\Phi=p^2\,\Phi \tag{7}$$

$$\Delta\Psi=\frac{p^2}{k^2}\Psi \tag{8}$$

$$q\,e^{-p|x|/\beta}=(1-2k^2)\,\Phi+2k^2\frac{\partial w_z}{\partial z}\quad\text{for } z=0 \tag{9}$$

$$0=\Psi-2\frac{\partial w_x}{\partial z}\quad\text{for } z=0. \tag{10}$$

Solution of Eqs. (6)–(8)

Excluding waves propagating in negative z-direction, the following general solutions of Eqs. (7) and (8) are obtained:

$$\Phi=\int_{-\infty}^{+\infty}A\,e^{-z\sqrt{p^2+\alpha^2}}\cos\alpha x\,d\alpha \tag{11}$$

$$\Psi=\int_{-\infty}^{+\infty}B\,e^{-z\sqrt{\frac{p^2}{k^2}+\alpha^2}}\sin\alpha x\,d\alpha, \tag{12}$$

where A and B are functions of p and α, only.

After insertion of the general solution in Eqs. (9) and (10), and making use of the inversion theorem for Fourier integrals, one obtains the following equations for determination of A and B:

$$\left(\frac{p^2}{2\,k^2}+\alpha^2\right)A+k^2\,\alpha\sqrt{\frac{p^2}{k^2}+\alpha^2}\,B=-\frac{q}{2\,\pi\,k^2\,\varrho\,c_d^2}\cdot\frac{\beta\,p^3}{p^2+\alpha^2\,\beta^2}$$

$$\alpha\sqrt{p^2+\alpha^2}\,A+k^2\left(\frac{p^2}{2\,k^2}+\alpha^2\right)B=0.$$

Solution of these equations yields

$$A=-\frac{q}{2\,\pi\,k^2\,\varrho\,c_d^2}\cdot\frac{\beta\,p^3}{p^2+\alpha^2\,\beta^2}\cdot\frac{\dfrac{p^2}{2\,k^2}+\alpha^2}{\left(\dfrac{p^2}{2\,k^2}+\alpha^2\right)^2-\alpha^2\sqrt{p^2+\alpha^2}\sqrt{\dfrac{p^2}{k^2}+\alpha^2}} \tag{13}$$

$$B=\frac{q}{2\,\pi\,k^2\,\varrho\,c_d^2}\cdot\frac{\beta\,p^3}{p^2+\alpha^2\,\beta^2}\cdot\frac{\dfrac{1}{k^2}\,\alpha\sqrt{p^2+\alpha^2}}{\left(\dfrac{p^2}{2\,k^2}+\alpha^2\right)^2-\alpha^2\sqrt{p^2+\alpha^2}\sqrt{\dfrac{p^2}{k^2}+\alpha^2}}. \tag{14}$$

From Eq. (6) we get for $z=0$:

$$p^2\,w_z=\int\limits_{-\infty}^{+\infty}\left[-\sqrt{p^2+\alpha^2}\,A-k^2\,\alpha\,B\right]\cos\alpha\,x\,d\,\alpha=$$

$$=\frac{q}{2\,\pi\,k^4\,\varrho\,c_d^2}\cdot\int\limits_0^{\infty}\frac{\beta\,p^3}{p^2+\alpha^2\,\beta^2}\cdot\frac{p^2\sqrt{p^2+\alpha^2}}{\left(\dfrac{p^2}{2\,k^2}+\alpha^2\right)^2-\alpha^2\sqrt{p^2+\alpha^2}\sqrt{\dfrac{p^2}{k^2}+\alpha^2}}\cos\alpha\,|x|\,d\,\alpha.$$

Confining ourselves to real, positive values of p, we get after putting

$$i\,\alpha\,|x|=-p\,\theta$$

the following expression

$$p^2\,w_z=-\frac{q}{2\,\pi\,k^4\,\varrho\,c_d^2\,|x|}\,\mathrm{Re}\,p\int\limits_0^{-i\infty}\frac{\beta}{1-\beta^2\dfrac{\theta^2}{x^2}}\cdot F\left(\frac{\theta^2}{x^2}\right)e^{-p\theta}\,d\,\theta, \tag{15}$$

where
$$F\left(\frac{\theta^2}{x^2}\right)=\frac{\sqrt{\dfrac{\theta^2}{x^2}-1}}{\left(\dfrac{1}{2\,k^2}-\dfrac{\theta^2}{x^2}\right)^2+\dfrac{\theta^2}{x^2}\sqrt{1-\dfrac{\theta^2}{x^2}}\sqrt{\dfrac{1}{k^2}-\dfrac{\theta^2}{x^2}}}.$$

The integrand of the integral in Eq. (15) has the branch points $\theta=\pm|x|$ and $\theta=\pm|x|/k$ and four simple poles, $\theta=\pm|x|/\beta$ and $\theta=\pm\theta_0$, where $\theta_0>|x|/k$. Thus the path of integration may be deformed to the path shown in Fig. 4.

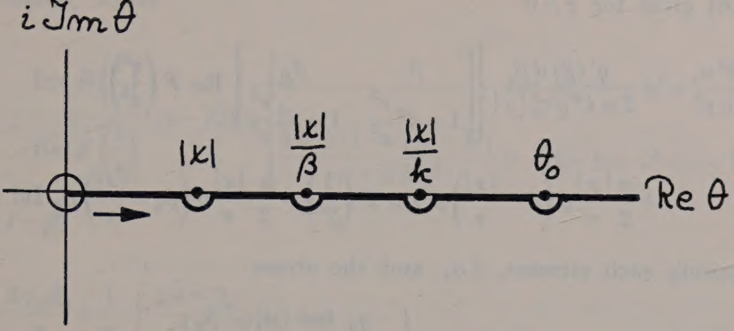


Fig. 4. Path of integration in the θ-plane.

Thus we obtain

$$p^2 w_z = -\frac{q}{2\pi k^4 \varrho c_d^2 |x|}\cdot\left\{p\int_0^\infty \frac{\beta}{1-\beta^2\frac{\theta^2}{x^2}} \operatorname{Re} F\left(\frac{\theta^2}{x^2}\right) e^{-p\theta}\, d\theta -\right.$$

$$\left. -\operatorname{Re} p\cdot\pi i\cdot\frac{|x|}{2} F\left(\frac{1}{\beta^2}\right) e^{-p|x|/\beta}\right\},$$

where the integral is to be taken in the sense of the Cauchy principal value, and the proper branch of the function F is found by studying Fig. 4. Inverting the Laplace-transform, we get for $\tau > 0$:

$$\frac{\partial^2 u_z}{\partial\tau^2} = -\frac{q}{2\pi k^2 \varrho c_d^2 |x|}\left\{\frac{\beta}{1-\beta^2\frac{\tau^2}{x^2}} \operatorname{Re} F\left(\frac{\tau^2}{x^2}\right) + \frac{\pi}{2}\frac{|x|}{\tau}\beta \operatorname{Im} F\left(\frac{1}{\beta^2}\right)\delta\left(\beta-\frac{|x|}{\tau}\right)\right\}. \quad (16)$$

Integration of this expression in order to find u_z is not necessary for solution of Problem C.

The method used for inverting the Laplace transform was introduced by Cagniard (1939).

Problem C

A pressure q_0 is acting on the infinite strip $|x| < v_0 t$ of the surface $z=0$ of a semi-infinite solid $z \geqslant 0$. The displacement u_z at the surface $z=0$ is zero for $|x| \geqslant v_0 t$. Find the normal stress σ_z at the surface $z=0$ for $|x| > v_0 t$.

Mathematical statement of the problem

If the stress on the surface is

$$d\sigma_z = \begin{cases} 0 & \text{for} \quad |x| < \beta_0 \tau \\ -q'(\beta)\, d\beta & \text{for } \beta_0\tau < |x| < \beta\tau \\ 0 & \text{for } \beta\tau < |x|. \end{cases}$$

Eq. (16) gives for $\tau > 0$

$$\frac{\partial^2 u_z}{\partial \tau^2} = -\frac{q'(\beta)\,d\beta}{2\pi k^4 \varrho c_d^2 |x|} \left\{ \left[\frac{\beta}{1-\beta^2 \frac{\tau^2}{x^2}} - \frac{\beta_0}{1-\beta_0^2 \frac{\tau^2}{x^2}} \right] \operatorname{Re} F\left(\frac{\tau^2}{x^2}\right) + \right.$$

$$\left. + \frac{\pi}{2} \frac{|x|}{\tau} \delta\left(\beta - \frac{|x|}{\tau}\right) \beta \operatorname{Im} F\left(\frac{1}{\beta^2}\right) - \frac{\pi}{2} \cdot \frac{|x|}{\tau} \delta\left(\beta_0 - \frac{|x|}{\tau}\right) \beta_0 \operatorname{Im} F\left(\frac{1}{\beta_0^2}\right) \right\}.$$

Superposing such stresses, $d\sigma_z$, and the stress

$$\sigma_z = \begin{cases} -q_0 & \text{for } |x| < \beta_0 \tau \\ 0 & \text{for } \beta_0 \tau < |x| \end{cases}$$

on the surface, in such a way that $\partial^2 u_z / \partial \tau^2 = 0$ for $|x| > \beta_0 \tau$, we get the integral equation

$$\int_{\beta_0}^{1} \left[\frac{\beta}{1-\beta^2 \frac{\tau^2}{x^2}} - \frac{\beta_0}{1-\beta_0^2 \frac{\tau^2}{x^2}} \right] q'(\beta)\,d\beta \cdot \operatorname{Re} F\left(\frac{\tau^2}{x^2}\right) +$$

$$+ \frac{\pi}{2} \int_{\beta_0}^{1} \delta\left(\beta - \frac{|x|}{\tau}\right) \cdot \beta \cdot \operatorname{Im} F\left(\frac{1}{\beta^2}\right) \cdot \frac{|x|}{\tau} q'(\beta)\,d\beta +$$

$$+ \frac{q_0 \beta_0}{1-\beta_0^2 \frac{\tau^2}{x^2}} \operatorname{Re} F\left(\frac{\tau^2}{x^2}\right) = 0 \quad \text{for } \beta_0 \tau < |x| < \tau.$$

Putting $x^2/\tau^2 = r$, $\beta^2 = s$ and $q'(\beta) = \varphi(s)$, we obtain for $\beta_0^2 < r < 1$

$$-\pi \frac{\operatorname{Im} F\left(\frac{1}{r}\right)}{\operatorname{Re} F\left(\frac{1}{r}\right)} (r-\beta_0^2)\,\varphi(r) + \int_{\beta_0^2}^{1} \frac{\varphi(s)}{s-r} \frac{(r-\beta_0^2)\sqrt{s} - \beta_0 (r-s)}{\sqrt{s}}\,ds = 2 q_0 \beta_0$$

or

$$-\frac{\operatorname{Im} F\left(\frac{1}{r}\right)}{\operatorname{Re} F\left(\frac{1}{r}\right)} (r-\beta_0^2)\,\varphi(r) + \frac{1}{\pi} \int_{\beta_0^2}^{1} \frac{(s-\beta_0^2)\,\varphi(s)}{s-r}\,ds - \frac{1}{\pi} \int_{\beta_0^2}^{1} \frac{\sqrt{s}-\beta_0}{\sqrt{s}} \varphi(s)\,ds = \frac{2 q_0 \beta_0}{\pi}. \tag{17}$$

Thus a singular integral equation

$$a(r)\,\varphi(r) + \frac{1}{\pi} \int_{\beta_0^2}^{1} \frac{b(s)\,\varphi(s)}{s-r}\,ds = K \tag{18}$$

has been obtained, where

$$a(r) = -\frac{\operatorname{Im} F\left(\frac{1}{r}\right)}{\operatorname{Re} F\left(\frac{1}{r}\right)} \cdot (r-\beta_0^2) = \begin{cases} 0 & \text{for } \beta_0^2 < r < k^2 \\ \dfrac{4k^3\sqrt{(1-r)(r-k^2)}}{(r-2k^2)^2}(r-\beta_0^2) & \text{for } k^2 < r < 1 \end{cases}$$

$$b(r) = r - \beta_0^2$$

$$K = \frac{2q_0\beta_0}{\pi} + \frac{1}{\pi}\int_{\beta_0^2}^{1} \frac{\sqrt{s}-\beta_0}{\sqrt{s}}\,\varphi(s)\,ds.$$

Integral equations of similar type have been studied by Carleman (1922). Further studies are reported in the monograph written by Muskhelishvili (1953).

Solution of Eq. (18)

The general solution of Eq. (18) can be written (cf. Muskhelishvili (1953)).

$$\varphi(r) = K\cdot\frac{a(r)}{a^2(r)+b^2(r)} - K\cdot\frac{e^{-\omega(r)}}{\pi(r-\beta_0^2)\sqrt{a^2(r)+b^2(r)}}\int_{\beta_0^2}^{1}\frac{(s-\beta_0^2)\,e^{\omega(s)}\,b(s)}{(s-r)\sqrt{a^2(s)+b^2(s)}}\,ds + {} \\ + L\cdot\frac{e^{-\omega(r)}}{(r-\beta_0^2)\sqrt{a^2(r)+b^2(r)}}, \tag{19}$$

where L is an arbitrary constant and

$$\omega(r) = \int_{\beta_0^2}^{1}\frac{\Theta(s)}{s-r}\,ds \qquad \Theta(r) = \frac{1}{2\pi i}\,ln\,\frac{a(r)+ib(r)}{a(r)-ib(r)}. \tag{20}$$

$\omega(r)$ has been calculated (see Appendix 1) and one obtains

$$e^{\omega(r)} = \begin{cases} \sqrt{\dfrac{1-r}{r-\beta_0^2}}\cdot\dfrac{r(r-k_s^2)}{(r-2k^2)^2-4k^3\sqrt{(1-r)(k^2-r)}} & \text{for } \beta_0^2 < r < k^2 \\ \sqrt{\dfrac{1-r}{r-\beta_0^2}}\cdot\dfrac{r(r-k_s^2)}{\sqrt{(r-2k^2)^4+16k^6(1-r)(r-k^2)}} & \text{for } k^2 < r < 1 \end{cases}.$$

The next step is to calculate the integral in Eq. (19). One obtains (see Appendix 2):

$$\int_{\beta_0^2}^{1} \frac{(s-\beta_0^2)\, e^{\omega(s)}\, b(s)}{(s-r)\sqrt{a^2(s)+b^2(s)}}\, ds =$$

$$= \begin{cases} \pi(M-r) & \text{for} \quad \beta_0^2 < r < k^2 \\ \pi(M-r) + \pi \cdot 4\,k^3 \dfrac{r\,(r-k_s^2)(1-r)\sqrt{(r-k^2)(r-\beta_0^2)}}{(r-2\,k^2)^4 + 16\,k^6 (1-r)(r-k^2)} & \text{for} \quad k^2 < r < 1, \end{cases}$$

where M is a constant that is not necessary to calculate.

Thus one obtains the following solution of Eq. (18):

For $\beta_0^2 < r < k^2$:

$$\varphi(r) = -K \cdot \frac{(M-r)\left[(r-2\,k^2)^2 - 4\,k^3\sqrt{(1-r)(k^2-r)}\right]}{r\,(r-k_s^2)\sqrt{(1-r)(r-\beta_0^2)^3}} +$$

$$+ L \cdot \frac{(r-2\,k^2)^2 - 4\,k^2\sqrt{(1-r)(k^2-r)}}{r\,(r-k_s^2)\sqrt{(1-r)(r-\beta_0^2)^3}}.$$

For $k^2 < r < 1$:

$$\varphi(r) = -K \cdot \frac{(M-r)(r-2\,k^2)^2}{r\,(r-k_s^2)\sqrt{(1-r)(r-\beta_0^2)^3}} + L \cdot \frac{(r-2\,k^2)^2}{r\,(r-k_s^2)\sqrt{(1-r)(r-\beta_0^2)^3}}.$$

This can be written in the following way:

For $\beta_0^2 < r < k^2$:

$$\varphi(r) = Q \cdot \frac{(r-2\,k^2)^2 - 4\,k^3\sqrt{(1-r)(k^2-r)}}{r\sqrt{(1-r)(r-\beta_0^2)^3}} + Q_1 \cdot \frac{(r-2\,k^2)^2 - 4\,k^3\sqrt{(1-r)(k^2-r)}}{(r-k_s^2)\sqrt{(1-r)(r-\beta_0^2)^3}}. \quad (21)$$

For $k^2 < r < 1$:

$$\varphi(r) = Q \cdot \frac{(r-2\,k^2)^2}{r\sqrt{(1-r)(r-\beta_0^2)^3}} + Q_1 \cdot \frac{(r-2\,k^2)^2}{(r-k_s^2)\sqrt{(1-r)(r-\beta_0^2)^3}}, \quad (22)$$

where Q and Q_1 are constants.

It is possible to show (see Problem D and Appendix 3) that the problem is satisfied by the term containing Q, only. Thus the solution is

$$\varphi(r) = \begin{cases} Q \cdot \dfrac{(r-2\,k^2)^2 - 4\,k^3\sqrt{(1-r)(k^2-r)}}{r\sqrt{(1-r)(r-\beta_0^2)^3}} & \text{for} \quad \beta_0^2 < r < k^2 \quad (23) \\ Q \cdot \dfrac{(r-2\,k^2)^2}{r\sqrt{(1-r)(r-\beta_0^2)^3}} & \text{for} \quad k^2 < r < 1, \quad (24) \end{cases}$$

where Q is to be determined from the equation

$$Q - \frac{1}{\pi}\int_{\beta_0^2}^{1} \frac{\sqrt{s}-\beta_0}{\sqrt{s}}\,\varphi(s)\,ds = \frac{2\,q_0\,\beta_0}{\pi}.$$

One obtains (see Appendix 4):

$$Q = q_0\frac{\beta_0^2(1-\beta_0^2)}{g(\beta_0)},$$

where

$$g(\beta_0) = [(1-4k^2)\beta_0^2 + 4k^4]\,\mathbf{K}(\sqrt{1-\beta_0^2}) - \frac{1}{\beta_0^2}[\beta_0^4 - 4k^2(1+k^2)\beta_0^2 + 8k^4]\,\mathbf{E}(\sqrt{1-\beta_0^2}) -$$

$$-4k^2(1-\beta_0^2)\,\mathbf{K}\left(\sqrt{1-\left(\frac{\beta_0}{k}\right)^2}\right) + \frac{8k^4}{\beta_0^2}(1-\beta_0^2)\,\mathbf{E}\left(\sqrt{1-\left(\frac{\beta_0}{k}\right)^2}\right), \qquad (25)$$

where **K** and **E** are the complete elliptic integrals of the first and the second kind, respectively.

The stress $\sigma_z(x)$ at the surface $z=0$

The stress $\sigma_z(x)$ on the surface $z=0$ for $|x| > \beta_0\tau$ is given by

$$\sigma_z(x) = -\int_{|x|/\tau}^{1} q'(\beta)\,d\beta = -\frac{1}{2}\int_{x^2/\tau^2}^{1}\frac{\varphi(r)}{\sqrt{r}}\,dr. \qquad (26)$$

Writing $|x|/\tau = \xi$, one obtains (see Appendix 5):

$$\sigma_z(\xi) = \frac{q_0}{g(\beta_0)}\Bigg\{\frac{1}{\xi\sqrt{\xi^2-\beta_0^2}}[4k^3(1-\beta_0^2)\sqrt{k^2-\xi^2} - (\beta_0^2-2k^2)^2\sqrt{1-\xi^2}] +$$

$$+[(1-4k^2)\beta_0^2+4k^4]\,F\left(\arcsin\sqrt{1-\beta_0^2},\ \arcsin\sqrt{\frac{\xi^2-\beta_0^2}{\xi^2(1-\beta_0^2)}}\right) -$$

$$-\frac{1}{\beta_0^2}[\beta_0^4-4k^2(1+k^2)\beta_0^2+8k^4]\,E\left(\arcsin\sqrt{1-\beta_0^2},\ \arcsin\sqrt{\frac{\xi^2-\beta_0^2}{\xi^2(1-\beta_0^2)}}\right) -$$

$$-4k^2(1-\beta_0^2)\,F\left(\arcsin\sqrt{1-\left(\frac{\beta_0}{k}\right)^2},\ \arcsin\sqrt{\frac{k^2(\xi^2-\beta_0^2)}{\xi^2(k^2-\beta_0^2)}}\right) +$$

$$+\frac{8k^4}{\beta_0^2}(1-\beta_0^2)\,E\left(\arcsin\sqrt{1-\left(\frac{\beta_0}{k}\right)^2},\ \arcsin\sqrt{\frac{k^2(\xi^2-\beta_0^2)}{\xi^2(k^2-\beta_0^2)}}\right)\Bigg\} - q_0 \qquad (27)$$

for $\beta_0 < \xi < k$, and

$$\sigma_z(\xi) = \frac{q_0}{g(\beta_0)} \left\{ -\frac{(\beta_0^2 - 2k^2)^2 \sqrt{1-\xi^2}}{\xi \sqrt{\xi^2 - \beta_0^2}} + \right.$$

$$+ [(1-4k^2)\beta_0^2 + 4k^4]\, F\left(\arcsin \sqrt{1-\beta_0^2},\ \arcsin \sqrt{\frac{\xi^2 - \beta_0^2}{\xi^2(1-\beta_0^2)}}\right) -$$

$$\left. - \frac{1}{\beta_0^2}[\beta_0^4 - 4k^2(1+k^2)\beta_0^2 + 8k^4]\, E\left(\arcsin \sqrt{1-\beta_0^2},\ \arcsin \sqrt{\frac{\xi^2 - \beta_0^2}{\xi^2(1-\beta_0^2)}}\right)\right\} -$$

$$- q_0 \cdot \frac{g_1(\beta_0)}{g(\beta_0)} \tag{28}$$

for $k < \xi < 1$. There

$$g_1(\beta_0) = [(1-4k^2)\beta_0^2 + 4k^4]\, \mathbf{K}(\sqrt{1-\beta_0^2}) - (1/\beta_0^2)[\beta_0^4 - 4k^4(1+k^2)\beta_0^2 + 8k^4]\, \mathbf{E}(\sqrt{1-\beta_0^2})$$

and F and E are elliptic integrals of the first and the second kind, respectively.
Further

$$\sigma_z(\xi) = -q_0 \quad \text{for} \quad 0 < \xi < \beta_0 \tag{29}$$

and

$$\sigma_z(\xi) = 0 \quad \text{for} \quad 1 < \xi. \tag{30}$$

For $\beta_0 \ll 1$, $\xi \ll 1$ one obtains

$$\sigma_z(\xi) = q_0\left(\frac{\xi}{\sqrt{\xi^2 - \beta_0^2}} - 1\right) \quad \text{i.e.} \quad \sigma_z(x) = q_0\left(\frac{|x|}{\sqrt{x^2 - (v_0 t)^2}} - 1\right)$$

in agreement with the well-known static solution, if $2v_0 t$ is identified with the width of the uniformly loaded strip.

The stress σ_z at the surface $z=0$ has thus been completely determined by the expressions (27)–(30). An alternative expression, suitable for certain uses, is given by:

$$\sigma_z = -\frac{4k^4 Q}{|x|} \cdot \operatorname{Re} \int_{\Gamma(\tau)} \frac{\left(\frac{1}{2k^2} - \frac{\tau^2}{x^2}\right)^2 - \frac{\tau^2}{x^2}\sqrt{\left(\frac{\tau^2}{x^2} - 1\right)\left(\frac{\tau^2}{x^2} - \frac{1}{k^2}\right)}}{\sqrt{\left(\frac{\tau^2}{x^2} - 1\right)\left(1 - \beta_0^2 \frac{\tau^2}{x^2}\right)^3}}\, d\tau, \tag{31}$$

where $\Gamma(\tau) = \Gamma_1 + \gamma + \Gamma_2(\tau)$ is the path of integration shown in Fig. 5. The proper branches of the integrand are found by making indentations below the points $\tau = |x|$ and $\tau = |x|/k$.

It can be shown if γ is sufficiently small that the expression (31) expresses σ_z for all values of x, except for the neighbourhood of $|x| = \beta_0 \tau$ (Appendix 6).

Problem D

A pressure q_0 is acting on the infinite strip $|x| < v_0 t$ of the surface $z=0$ of a semi-infinite solid $z \geqslant 0$. The displacement, u_z, at the surface $z=0$ is zero for $|z| \geqslant v_0 t$. Find the displacement u_z at the surface $z=0$ for $|x| < v_0 t$.

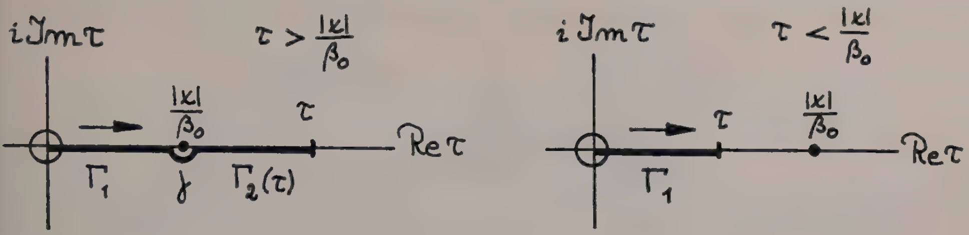


Fig. 5. Path of integration in the τ-plane.

The displacement u_z at the surface $z=0$

We may use the expression (31) for the stress σ_z on the surface. Thus the problem is no longer a mixed boundary value problem and the solution will be completely analogous to the solution of Problem B. We need to calculate the Laplace transform of σ_z.

Supposing γ (see Fig. 5) to be a semi-circle with radius $\varepsilon\cdot|x|$, we get

$$\sigma_z=-\frac{4\,k^4\,Q}{|x|}\operatorname{Re}\int\limits_0^{\tau} f(\tau)\cdot h(\tau)\,d\tau-\frac{8\,k^4\,Q}{|x|}\cdot Re\,I,$$

where

$$f(\tau)=\frac{\left(\dfrac{1}{2\,k^2}-\dfrac{\tau^2}{x^2}\right)^2-\dfrac{\tau^2}{x^2}\sqrt{\left(\dfrac{\tau^2}{x^2}-1\right)\left(\dfrac{\tau^2}{x^2}-\dfrac{1}{k^2}\right)}}{\sqrt{\left(\dfrac{\tau^2}{x^2}-1\right)\left(1-\beta_0^2\dfrac{\tau^2}{x^2}\right)^3}}$$

$$h(\tau)=1-U\left(\tau-\frac{|x|}{\beta_0}+\varepsilon\cdot|x|\right)+U\left(\tau-\frac{|x|}{\beta_0}-\varepsilon\cdot|x|\right)$$

and

$$I=\int\limits_{\gamma} f(\tau)\,d\tau.$$

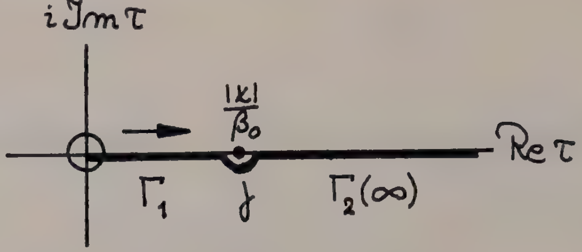
Writing $S_z=p\int\limits_0^{\infty}e^{-p\tau}\,\sigma_z\,d\tau$, we obtain for real, positive values of p:

$$\frac{|x|}{4\,k^4\,Q}S_z=-\operatorname{Re}p\int\limits_0^{\infty}e^{-p\tau}\int\limits_0^{\tau}f(\tau')\,h(\tau')\,d\tau'\,d\tau-\operatorname{Re}p\int\limits_0^{\infty}e^{-p\tau}\,I\,d\tau=$$

$$=-\operatorname{Re}\int\limits_0^{\infty}e^{-p\tau}f(\tau)\,h(\tau)\,d\tau-\operatorname{Re}p\int\limits_0^{\infty}e^{-p\tau}\,I\,d\tau=$$

$$=-\operatorname{Re}\int\limits_{\Gamma(\infty)}e^{-p\tau}f(\tau)\,d\tau+\operatorname{Re}\int\limits_{\gamma}e^{-p\tau}f(\tau)\,d\tau-\operatorname{Re}p\int\limits_0^{\infty}e^{-p\tau}\,I\,d\tau,$$

where $\Gamma(\infty)=\Gamma_1+\gamma+\Gamma_2(\infty)$ is the path of integration shown in Fig. 6.



Fig, 6. Path of integration in the τ-plane.

Thus, for $\varepsilon \ll 1$:

$$\frac{|x|}{4\,k^4\,Q} S_z = -\operatorname{Re} \int\limits_{\Gamma(\infty)} e^{-p\tau} f(\tau)\, d\tau + e^{-p|x|/\beta_0} \operatorname{Re} \int\limits_{\gamma} f(\tau)\, d\tau -$$

$$-\operatorname{Re} p \int\limits_{\gamma} f(\tau')\, d\tau' \cdot \int\limits_{|x|/\beta_0}^{\infty} e^{-p\tau}\, d\tau + O\,(p\,x^2\,\varepsilon^{\frac{1}{2}}) =$$

$$= -\operatorname{Re} \int\limits_{\Gamma(\infty)} e^{-p\tau} f(\tau)\, d\tau + O\,(p\,x^2\,\varepsilon^{\frac{1}{2}}),$$

where O indicates "the order of". As the inverse Laplace transform of the last term is

$$\delta\,(\tau)\, O\,(x^2\,\varepsilon^{\frac{1}{2}})$$

this term corresponds to a load

$$\sigma_z = O\,(4\,k^4\,Q\,|x| \cdot \varepsilon^{\frac{1}{2}}) \cdot \delta\,(\tau)$$

which certainly has a very small influence on the displacement u_z on the surface for $|x| < \beta_0\,\tau$, and this term can thus be neglected in the following. Moreover, in the limit $\varepsilon \to 0$, one obtains the exact Laplace transform of the stress $(\sigma_z)_{z=0}$ corresponding to the conditions in Problem D:

$$S_z = p \int\limits_0^{\infty} e^{-p\tau}\, \sigma_z\, d\tau = -\frac{4\,k^4\,Q}{|x|} \cdot \operatorname{Re} \int\limits_{\Gamma(\infty)} e^{-p\tau} f(\tau)\, d\tau.$$

In analogy with the solution of Problem B, we wish to invert Fourier integrals of the kind shown in Eq. (11). Therefore we first deform the path of integration:

$$S_z = -\frac{4\,k^4\,Q}{|x|} \operatorname{Re} \int\limits_0^{-i\infty} f(\tau)\, e^{-p\tau}\, d\tau.$$

Then, after putting $\tau = \alpha |x|/(i p)$, we obtain

$$S_z = -\frac{2 k^4 Q}{p} \int_{-\infty}^{+\infty} \frac{\left(\frac{p^2}{2 k^2} + \alpha^2\right)^2 - \alpha^2 \sqrt{(p^2 + \alpha^2)\left(\frac{p^2}{k^2} + \alpha^2\right)}}{\sqrt{p^2 + \alpha^2}\sqrt{(p^2 + \alpha^2 \beta_0^2)^3}} \cos \alpha |x| \, d\alpha . \tag{32}$$

The analogy with the solution of Problem B is then quite obvious. It is merely to replace the expression (5) by the expression (32), i.e. to replace the quantity

$$\frac{q}{2 \pi k^2 \varrho c_d^2} \cdot \frac{\beta p^3}{p^2 + \alpha^2 \beta^2}$$

occurring in Eqs. (13) and (14) by the quantity

$$\frac{k^2 Q p}{\varrho c_d^2} \cdot \frac{\left(\frac{p^2}{2 k^2} + \alpha^2\right)^2 - \alpha^2 \sqrt{(p^2 + \alpha^2)\left(\frac{p^2}{k^2} + \alpha^2\right)}}{\sqrt{p^2 + \alpha^2}\sqrt{(p^2 + \alpha^2 \beta^2)^3}} .$$

We obtain for $z = 0$:

$$p^2 w_z = \frac{Q}{\varrho c_d^2 |x|} \cdot \mathrm{Re}\, p \cdot \int_0^{-i\infty} \frac{i e^{-p\theta}}{\sqrt{\left(1 - \beta_0^2 \frac{\theta^2}{x^2}\right)^3}} d\theta . \tag{33}$$

After deforming the path of integration to the path shown in Fig. 6, we get

$$p^2 w_z = -\frac{Q}{\varrho c_d^2 |x|} \cdot p \int_0^{\infty} U\left(\theta - \frac{|x|}{\beta_0}\right) \frac{e^{-p\theta}}{\sqrt{\left(\beta_0^2 \frac{\theta^2}{x^2} - 1\right)^3}} d\theta \tag{34}$$

and thus
$$\frac{\partial^2 u_z}{\partial \tau^2} = -\frac{Q}{\varrho c_d^2 |x|} \cdot \frac{U\left(\tau - \frac{|x|}{\beta_0}\right)^3}{\sqrt{\left(\beta_0^2 \frac{\tau^2}{x^2} - 1\right)^3}} .$$

After two integrations we obtain

$$u_z = \frac{Q}{\varrho c_d^2 \beta_0} \sqrt{\tau^2 - \frac{x^2}{\beta_0^2}} \cdot U\left(\tau - \frac{|x|}{\beta_0}\right). \tag{35}$$

The shape of the surface displacement is thus a semi-ellips

$$\frac{x^2}{\beta_0^2 \tau^2} + \frac{\varrho^2 c_d^4 \beta_0^4}{Q^2} \cdot \frac{u_z^2}{\beta_0^2 \tau^2} = 1, \qquad u_z \geqslant 0. \tag{36}$$

If $\beta_0 \ll 1$, and $\beta_0 \tau = a$, one obtains, after making use of the relations

$$\frac{Q(\beta_0)}{\beta_0^2} \to \frac{q_0}{2k^2(1-k^2)} \quad \text{when} \quad \beta_0 \to 0$$

$$k^2 = \frac{1-2\nu}{2(1-\nu)}, \quad \text{where} \quad \nu = \text{Poisson's ratio}$$

$$\varrho c_d^2 = \frac{(1-\nu)E}{(1+\nu)(1-2\nu)}, \quad \text{where} \quad E = \text{Young's modulus}$$

the following equation
$$\frac{x^2}{a^2} + \frac{u_z^2}{\left[2(1-\nu^2)\frac{q_0}{E}a\right]^2} = 1$$

in agreement with the well-known static solution (cf. for instance Sneddon, 1946).

The solution of Problem D provides also a verification that the solution (23), (24) satisfies Problem C.

The stresses in the semi-infinite solid

Because of the conditions of plane strain, one obtains

$$\sigma_y = (1-2k^2)\,\varrho c_d^2 \phi \tag{37}$$

$$\frac{\sigma_x + \sigma_z}{2} = (1-k^2)\,\varrho c_d^2 \phi. \tag{38}$$

The state of stress is thus completely determined by the following three quantities:

$$\phi, \quad \sigma_z, \quad \tau_{xz}.$$

Let us start to determine ϕ. We obtain in analogy with Eq. (11):

$$\Phi = \int_{-\infty}^{+\infty} A\, e^{-z\sqrt{p^2+\alpha^2}} \cos \alpha x \, d\alpha = -\frac{2k^2 Q p}{\varrho c_d^2} \cdot \mathrm{Re} \int_0^\infty \frac{\left(\frac{p^2}{2k^2} + \alpha^2\right) e^{-z\sqrt{p^2+\alpha^2}+i|x|\alpha}}{\sqrt{p^2+\alpha^2}\,\sqrt{(p^2+\alpha^2\beta_0^2)^3}}\, d\alpha \cdot$$

Putting $\alpha = p\varkappa$, one obtains

$$\Phi = -\frac{2k^2 Q}{\varrho c_d^2} \cdot \mathrm{Re} \int_0^\infty \frac{\left(\frac{1}{2k^2} + \varkappa^2\right) e^{-p(z\sqrt{1+\varkappa^2}-i|x|\varkappa)}}{\sqrt{1+\varkappa^2}\,\sqrt{(1+\varkappa^2\beta_0^2)^3}}\, d\varkappa.$$

The path of integration can be deformed to the straight line from $\varkappa = 0$ to

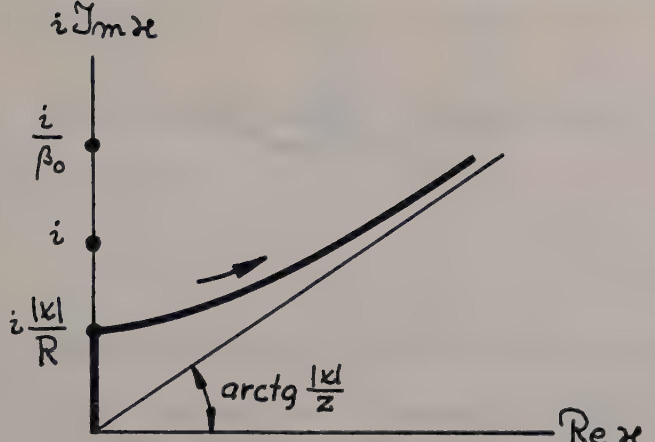


Fig. 7. Path of integration in the τ-plane.

$$\varkappa = i\frac{|x|}{\sqrt{x^2+z^2}} = i\frac{|x|}{R}$$

and that branch of the hyperbola

$$\frac{(\mathrm{Re}\,\varkappa)^2}{z^2} - \frac{(\mathrm{Im}\,\varkappa)^2}{x^2} = -\frac{1}{R^2}$$

for which $\mathrm{Re}\,\varkappa > 0$ and $\mathrm{Im}\,\varkappa > 0$. (See Fig. 7).

The real part of the integral along the straight line is zero and thus the path of integration is reduced to the hyperbola-branch. If then the substitution

$$\theta = z\sqrt{1+\varkappa^2} - i|x|\varkappa$$

is introduced, the path of integration in the θ-plane becomes the positive real axis from $\theta = \sqrt{x^2+z^2} = R$ to $\theta = \infty$.

One obtains
$$d\varkappa = \frac{\sqrt{1+\varkappa^2}}{\sqrt{\theta^2-R^2}}\,d\theta$$

and thus

$$\Phi = -\frac{2k^2Q}{\varrho c_d^2}\cdot \mathrm{Re}\int_0^\infty U(\theta - R)\cdot\frac{e^{-p\theta}\,d\theta}{\sqrt{\theta^2-R^2}}\cdot\left[\frac{\frac{1}{2k^2}+\varkappa^2}{\sqrt{(1+\varkappa^2\beta_0^2)^3}}\right]_{\varkappa=\frac{\sqrt{\theta^2-R^2}\,z+i\theta|x|}{R^2}}.$$

Then
$$\phi = -\frac{2k^2Q}{\varrho c_d^2}\,\mathrm{Re}\int_R^\tau \frac{d\theta}{\sqrt{\theta^2-R^2}}\cdot\left[\frac{\frac{1}{2k^2}+\varkappa^2}{\sqrt{(1+\varkappa^2\beta_0^2)^3}}\right]_{\varkappa=\frac{\sqrt{\theta^2-R^2}\,z+i\theta|x|}{R^2}}. \tag{39}$$

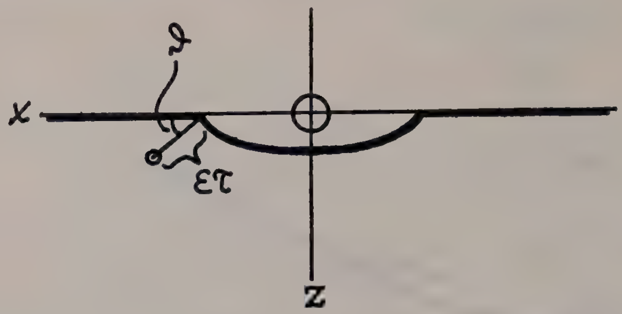


Fig. 8. Coordinate ε and ϑ in the vicinity of the point $|x| = \beta_0\tau$, $z = 0$.

Now we investigate the variation of the magnitude of ϕ at a small distance $\varepsilon\tau$ from the crack-tip with the angular location as given by the angle ϑ, where

$$|x| = \beta_0\tau + \varepsilon\tau\cos\vartheta, \qquad z = \varepsilon\tau\sin\vartheta. \tag{40}$$

See Fig. 8.

Insertion of (40) in Eq. (39) gives, after higher order terms of ε have been neglected, and the substitution $u = \theta/\tau$ has been introduced, the following expression for $\varepsilon \ll 1$;

$$\phi \approx \frac{2k^2 Q}{\varrho c_d^2} \operatorname{Re} \int\limits_{\beta_0+\varepsilon\cos\vartheta}^{1} \frac{du}{\sqrt{u^2-\beta_0^2}} \cdot \frac{\dfrac{u^2}{\beta_0^2} - \dfrac{1}{2k^2}}{\left[1 - u^2 + 2\dfrac{\varepsilon}{\beta_0} u^2 \cos\vartheta + 2i\dfrac{\varepsilon}{\beta_0} u\sqrt{u^2-\beta_0^2}\sin\vartheta\right]^{\frac{3}{2}}}.$$

It is evident that the main contribution to this integral is given by values of u in the neighbourhood of $u = 1$, and thus we can write

$$\phi \approx \frac{2k^2 Q}{\varrho c_d^2} \cdot \frac{\dfrac{1}{\beta_0^2} - \dfrac{1}{2k^2}}{\sqrt{1-\beta_0^2}} \operatorname{Re} \int\limits_{\beta_0}^{1} \frac{du}{\left[1 - u^2\left(1 - 2\dfrac{\varepsilon}{\beta_0}\cos\vartheta - 2i\dfrac{\varepsilon}{\beta_0}\sqrt{1-\beta_0^2}\sin\vartheta\right)\right]^{\frac{3}{2}}}.$$

After integration we obtain

$$\phi \approx \frac{Q}{\varrho c_d^2} \cdot \frac{2k^2 - \beta_0^2}{\beta_0^2\sqrt{1-\beta_0^2}} \operatorname{Re} \frac{1}{\left[2\dfrac{\varepsilon}{\beta_0}\cos\vartheta + 2i\dfrac{\varepsilon}{\beta_0}\sqrt{1-\beta_0^2}\sin\vartheta\right]^{\frac{1}{2}}}$$

and thus

$$\phi \to \frac{Q}{2\varrho c_d^2} \cdot \frac{2k^2 - \beta_0^2}{\beta_0^2\sqrt{1-\beta_0^2}} \cdot \sqrt{\frac{\beta_0}{\varepsilon}} \sqrt{\frac{\sqrt{1-\beta_0^2\sin^2\vartheta} + \cos\vartheta}{1-\beta_0^2\sin^2\vartheta}} \tag{41}$$

when $\varepsilon \to 0$.

It is to be observed that when $\beta_0 \to 0$ Eq. (41) changes to

$$\phi \to \frac{q_0}{2\,\varrho\, c_d^2 (1-k^2)} \sqrt{\frac{2\,\beta_0}{\varepsilon}} \cos\frac{\vartheta}{2}, \qquad \varepsilon \ll 1$$

and thus, because of Eq. (38)

$$\frac{\sigma_x + \sigma_z}{2} \to q_0 \sqrt{\frac{\beta_0}{2\,\varepsilon}} \cos\frac{\vartheta}{2}, \qquad \varepsilon \ll 1 \tag{42}$$

in agreement with the static solution (Sneddon 1946).

σ_z and τ_{xz} are determined in a similar way as that for ϕ. One obtains

$$S_z = -\frac{2\,k^4\,Q}{p} \cdot \int\limits_{-\infty}^{+\infty} \frac{\left(\frac{p^2}{2\,k^2} + \alpha^2\right)^2 e^{-z\sqrt{p^2+\alpha^2}} - \alpha^2 \sqrt{p^2+\alpha^2} \sqrt{\frac{p^2}{k^2} + \alpha^2}\, e^{-z\sqrt{\frac{p^2}{k^2}+\alpha^2}}}{\sqrt{p^2+\alpha^2}\, \sqrt{(p^2+\alpha^2\beta_0^2)^3}} \cos\alpha x\, d\alpha$$

$$T_{xz} = -\frac{2\,k^4\,Q}{p} \cdot \int\limits_{-\infty}^{+\infty} \frac{\alpha\left(\frac{p^2}{2\,k^2} + \alpha^2\right)}{\sqrt{(p^2+\alpha^2\beta_0^2)^3}} \left[e^{-z\sqrt{p^2+\alpha^2}} - e^{-z\sqrt{\frac{p^2}{k^2}+\alpha^2}}\right] \sin\alpha x\, d\alpha,$$

where S_z and T_{xz} are the Laplace transforms of σ_z and τ_{xz}, respectively. S_z and T_{xz} contain two parts, one irrotational (characterized by $e^{-z\sqrt{p^2+\alpha^2}}$) and one equivoluminal (characterized by $e^{-z\sqrt{p^2/k^2+\alpha^2}}$). After the substitution $\alpha = p\varkappa$, the path of integration is deformed to a straight line along the imaginary axis and a hyperbola-branch in analogy with the case when ϕ was determined. After the substitution

$$\theta = z\sqrt{\frac{1}{k^2} + \varkappa^2} - i\,|x|\,\varkappa$$

in the equivoluminal part, the path of integration in the θ-plane becomes as before an infinite segment of the positive real axis. One obtains

$$\begin{aligned}\sigma_z = &-4\,k^4\,Q \cdot \mathrm{Re} \int\limits_{R}^{\tau} \frac{d\theta}{\sqrt{\theta^2 - R^2}} \left[\frac{\left(\frac{1}{2\,k^2} + \varkappa^2\right)^2}{\sqrt{(1+\varkappa^2\beta_0^2)^3}}\right]_{\varkappa = \frac{\sqrt{\theta^2 - R^2}\, z + i\,\theta\,|x|}{R^2}} + \\ &+ 4\,k^4\,Q \cdot \mathrm{Re} \int\limits_{R/k}^{\tau} \frac{d\theta}{\sqrt{\theta^2 - \frac{R^2}{k^2}}} \cdot \left[\frac{\varkappa^2\left(\frac{1}{2\,k^2} + \varkappa^2\right)}{\sqrt{(1+\varkappa^2\beta_0^2)^3}}\right]_{\varkappa = \frac{\sqrt{\theta^2 - \frac{R^2}{k^2}}\, z + i\,\theta\,|x|}{R^2}}\end{aligned} \tag{43}$$

and

$$\tau_{xz} = -\frac{x}{|x|}\cdot 4\,k^4\,Q\cdot \mathrm{Re}\int_R^\tau \frac{d\theta}{\sqrt{\theta^2-R^2}}\left[\frac{\varkappa\sqrt{1+\varkappa^2}\left(\frac{1}{2\,k^2}+\varkappa^2\right)}{i\sqrt{(1+\varkappa^2\beta_0^2)^2}}\right]_{\varkappa=\frac{\sqrt{\theta^2-R^2}\,z+i\theta|x|}{R^2}} + \qquad (44)$$

$$+\frac{x}{|x|}\cdot 4\,k^4\,Q\;\mathrm{Re}\int_{\frac{R}{k}}^\tau \frac{d\theta}{\sqrt{\theta^2-\frac{R^2}{k^2}}}\left[\frac{\varkappa\sqrt{\frac{1}{k^2}+\varkappa^2}\left(\frac{1}{2\,k^2}+\varkappa^2\right)}{i\sqrt{(1+\varkappa^2\,\beta_0^2)^3}}\right]_{\varkappa=\frac{\sqrt{\theta^2-\frac{R^2}{k^2}}\,z+i\theta|x|}{R^2}}.$$

It may be observed that no head-wave appears. This fact has a simple physical explanation: The surface $z=0$ is here prevented to move in the z-direction when $|x|>\beta_0\tau$, and thus also rotation of the elements at this part of the surface is prevented.

In the vicinity of the point $|x|=\beta_0\tau$, $z=0$, Eqs. (43) and (44) reduce to

$$\sigma_z \to \frac{Q}{\beta_0^4}\sqrt{\frac{\beta_0}{\varepsilon}}\left\{-\frac{(2\,k^2-\beta_0^2)^2}{2\sqrt{1-\beta_0^2}}\sqrt{\frac{\sqrt{1-\beta_0^2\sin^2\vartheta}+\cos\vartheta}{1-\beta_0^2\sin^2\vartheta}}\right.$$
$$\left.+\,2\,k^3\sqrt{k^2-\beta_0^2}\sqrt{\frac{\sqrt{1-(\beta_0^2/k^2)\sin^2\vartheta}+\cos\vartheta}{1-(\beta_0^2/k^2)\sin^2\vartheta}}\right\} \qquad (45)$$

$$\tau_{xz} \to \frac{x}{|x|}\cdot\frac{k^2 Q}{\beta_0^2}\cdot\sqrt{\frac{\beta_0}{\varepsilon}}\cdot\frac{2\,k^2-\beta_0^2}{\beta_0^2}\left\{\sqrt{\frac{\sqrt{1-\beta_0^2\sin^2\vartheta}-\cos\vartheta}{1-\beta_0^2\sin^2\vartheta}}\;-\right.$$
$$\left.-\sqrt{\frac{\sqrt{1-(\beta_0^2/k^2)\sin^2\vartheta}-\cos\vartheta}{1-(\beta_0^2/k^2)\sin^2\vartheta}}\right\}, \qquad (46)$$

where ε and ϑ are given by Eqs. (40).

When $\beta_0\to 0$, Eqs. (45) and (46) reduce to

$$\sigma_z \to q_0\sqrt{\frac{\beta_0}{2\,\varepsilon}}\cos\frac{\vartheta}{2}\left(1+\sin^2\vartheta-\sin^2\frac{\vartheta}{2}\right),\qquad \varepsilon\ll 1 \qquad (47)$$

and

$$\tau_{xz} \to \frac{x}{|x|}\cdot q_0\sqrt{\frac{\beta_0}{2\,\varepsilon}}\sin\frac{\vartheta}{2}\left(\cos^2\frac{\vartheta}{2}-\sin^2\vartheta\right),\qquad \varepsilon\ll 1 \qquad (48)$$

in agreement with the static solution (Sneddon 1946).

Eqs. (39), (43) and (44) determine the state of stress completely.

Discussion of the propagation of the brittle crack

The velocity of a brittle crack

By superposing a state of plane strain characterized by $\sigma_z = q_0$, $\sigma_y = \sigma_x = (1 - 2k^2) q_0$ to the semi-infinite solid considered in the preceeding sections, and then putting two such semi-infinite solids in contact with each other, the solution (35) indicates the propagation of an elliptic crack. However, the propagation velocity, $v_0 = \beta_0 c_d$, of the crack-tip as given by Eq. (35) presupposes that the two semi-infinite solids are joined together by a joint resistant to any finite stress if the velocity $v_0 < c_s$, where $c_s = k_s c_d$ is the velocity of surface (Rayleigh) waves. This follows from Eq. (27), which expresses the stress in the joint (after q_0 has been added to the right member) because the term

$$\frac{1}{\xi \sqrt{\xi^2 - \beta_0^2}} \left[4 k^3 (1 - \beta_0^2) \sqrt{k^2 - \xi^2} - (\beta_0^2 - 2 k^2)^2 \sqrt{1 - \xi^2}\right] \tag{49}$$

becomes infinite for $\xi = \beta_0$, i.e. at the crack-tip, and further it is positive in the neighbourhood of $\xi = \beta_0$ if $\beta_0 < k_s$.

On the other hand, the term (49) is negative in the neighbourhood of $\xi = \beta_0$ if $k_s < \beta_0 < k$. This indicates that, if the propagation velocity of the crack, v_0, has some value between c_s and c_r, we must presuppose that the joint cannot resist any tensile stress at all.

Thus, if the joint has some finite strength, the velocity of the crack-tip will be precisely $v_c = c_s$. This could perhaps have been predicted intuitively. However, we have arrived at this conclusion by tacitly assuming that the material of the two semi-infinite solids possessed an infinite strength. If we now suppose that this material and the joint are of equal (finite) strength, we are obviously dealing with the original problem, the propagation of a two-dimensional brittle crack in an infinite elastic solid.

In this case the correspondence between Problem A (Problems C and D) and the original problem is complete only if the stress state in the neighbourhood of $|x| = \beta_0 \tau$ is such that fracturing occurs only in the plane $z = 0$. In the absence of a reliable fracture criterion for a brittle material, we choose to investigate the behaviour of the dilatation ϕ and the normal stress σ_ϑ for a hypothetic fracture direction, defined by the angle ϑ, given by Eqs. (40). σ_ϑ is given by the relation

$$\sigma_\vartheta = \frac{\sigma_x + \sigma_z}{2} - \frac{\sigma_x - \sigma_z}{2} \cos 2\vartheta - \frac{|x|}{x} \tau_{xz} \sin 2\vartheta .$$

From Eqs. (38), (41), (45) and (46) ϕ and σ_ϑ can be determined in the vicinity of the crack-tip. One obtains for $\varepsilon \ll 1$

$$\phi \approx \frac{Q}{2 \varrho c_d^2} \cdot \frac{2 k^2 - \beta_0^2}{\beta_0^2 \sqrt{1 - \beta_0^2}} \sqrt{\frac{\beta_0}{\varepsilon}} \cdot F_1 (\beta_0, \vartheta) \tag{50}$$

$$\sigma_\vartheta \approx \frac{Q}{\beta_0^2} \cdot \frac{(1-k^2)(2k^2-\beta_0^2)}{2\sqrt{1-\beta_0^2}} \cdot \sqrt{\frac{\beta_0}{\varepsilon}} \cdot F_1(\beta_0, \vartheta) \cdot$$

$$\cdot \left\{ 1 + \frac{k^2}{\beta_0^2(1-k^2)} \cos 2\vartheta \left[\frac{4k\sqrt{k^2-\beta_0^2}\sqrt{1-\beta_0^2}}{(2k^2-\beta_0^2)} \cdot \frac{F_1(\beta_0/k, \vartheta)}{F_1(\beta_0, \vartheta)} - (2-\beta_0^2) \right] - \right. \quad (51)$$

$$\left. - \frac{2k^2\sqrt{1-\beta_0^2}}{\beta_0^2(1-k^2)} \sin 2\vartheta \cdot \frac{F_2(\beta_0, \vartheta) - F_2(\beta_0/k, \vartheta)}{F_1(\beta_0, \vartheta)} \right\},$$

where

$$F_1(\beta_0, \vartheta) = \sqrt{\frac{\sqrt{1-\beta_0^2 \sin^2 \vartheta} + \cos \vartheta}{1-\beta_0^2 \sin^2 \vartheta}}$$

and

$$F_2(\beta_0, \vartheta) = \sqrt{\frac{\sqrt{1-\beta_0^2 \sin^2 \vartheta} - \cos \vartheta}{1-\beta_0^2 \sin^2 \vartheta}}.$$

By the aid of Eqs. (50) and (51) it can be shown that ϕ and σ_ϑ both are maximum for $\vartheta = 0$ if $\beta_0 \leqslant k_s$. If a certain magnitude of ϕ or σ_ϑ were decisive for the occurrence of fracture, then the fracture would follow the plane $z = 0$. However, a real material shows anisotropy as regards possible crack-surface directions (the crack-surface will follow lattice planes) and inhomogeneities as regards the local "atomic" strength. Thus, in a real material, the crack would no longer follow exactly the plane $z = 0$, but it would oscillate across this plane. The tendency for such oscillations seems to be rather pronounced because the angular maximum of ϕ and σ_ϑ is rather flat (ϕ and σ_ϑ behave approximately like $\cos \vartheta/2$). A few calculations based on specific values of k and β_0 indicate that this maximum will be somewhat flattened when β_0 increases.

If the crack oscillates across the plane $z = 0$, the component of the crack-tip velocity in the $|x|$-direction will be somewhat lower than c_s.

The shape of the running crack

Eq. (36) shows that the crack is elliptic. The ratio between the semi-minor and semi-major axes is

$$\frac{Q(\beta_0)}{\varrho\, c_d^2\, \beta_0^2}.$$

A calculation of this ratio in the case of a material with Poisson's ratio $= \frac{2}{7}$ has been carried out. The result is approximately

$$1.837 \frac{q_0}{E} \quad \text{for} \quad \beta_0 = 0$$

$$1.04 \frac{q_0}{E} \quad \text{for} \quad \beta_0 = k_s.$$

The running crack is thus more eccentric than a static crack subjected to the same stress.

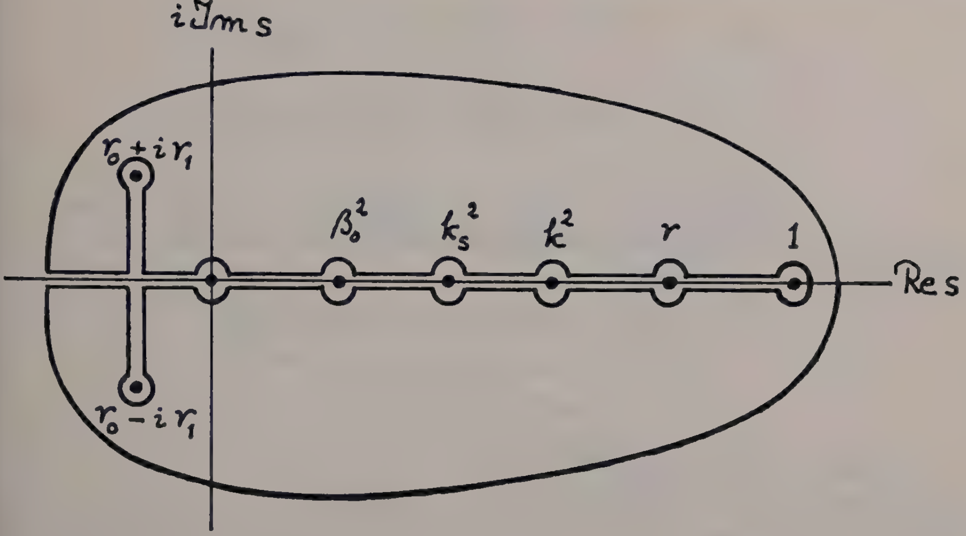


Fig. 9. Path of integration in the s-plane.

APPENDIX 1

Calculation of $\omega(r)$

From Eq. (20) we obtain

$$\Theta(r) = \begin{cases} \frac{1}{2} & \text{for } \beta_0^2 < r < k^2 \\ \frac{1}{2\pi i} \ln \frac{4k^3 \sqrt{(1-r)(r-k^2)} + i(r-2k^2)^2}{4k^2 \sqrt{(1-r)(r-k^2)} - i(r-2k^2)^2} & \text{for } k^2 < r < 1. \end{cases}$$

Thus
$$\omega(r) = \int_{\beta_0^2}^{1} \frac{\Theta(s)}{s-r} ds = \frac{1}{2} \int_{\beta_0^2}^{1} \frac{ds}{s-r} - \frac{1}{2\pi i} \cdot I,$$

where
$$I = \int_{k^2}^{1} \frac{1}{s-r} \ln \frac{(s-2k^2)^2 + i \cdot 4k^3 \sqrt{(1-s)(s-k^2)}}{(s-2k^2)^2 - i \cdot 4k^3 \sqrt{(1-s)(s-k^2)}} ds.$$

The integrand has the branch-points $s=1$, $s=k^2$, $s=0$, $s=k_s^2$, $s=r_0+ir_1$, $s=r_0-ir_1$, where the four last values of s are zeros to the equation

$$(r-2k^2)^4 + 16k^6(1-r)(r-k^2) = 0.$$

It can be shown that k_s is the ratio between the propagation velocities of surface (Rayleigh) waves and irrotational waves, and that $r_0 < 0$.

By choosing a path of integration shown in Fig. 9, one easily verifies that

$$I=\begin{cases} 2\pi i \ln \dfrac{(r-2k^2)^2-4k^3\sqrt{(1-r)\,(k^2-r)}}{r\,(r-k_s^2)} & \text{for } \beta_0^2<r<k^2 \\[2ex] 2\pi i \ln \dfrac{\sqrt{(r-2k^2)^4+16k^6(1-r)(k^2-r)}}{r\,(r-k_s^2)} & \text{for } k^2<r<1. \end{cases}$$

Then $$\omega(r)=\begin{cases} \ln\left(\sqrt{\dfrac{1-r}{r-\beta_0^2}}\cdot\dfrac{r\,(r-k_s^2)}{(r-2k^2)^2-4k^3\sqrt{(1-r)\,(k^2-r)}}\right) & \text{for } \beta_0^2<r<k^2 \\[2ex] \ln\left(\sqrt{\dfrac{1-r}{r-\beta_0^2}}\cdot\dfrac{r\,(r-k_s^2)}{\sqrt{(r-2k^2)^4+16k^6(1-r)(k^2-r)}}\right) & \text{for } k^2<r<1. \end{cases}$$

APPENDIX 2

Calculation of $\displaystyle\int_{\beta_0^2}^{1}\frac{(s-\beta_0^2)\,e^{\omega(s)}\,b(s)}{(s-r)\sqrt{a^2(s)+b^2(s)}}\,ds.$

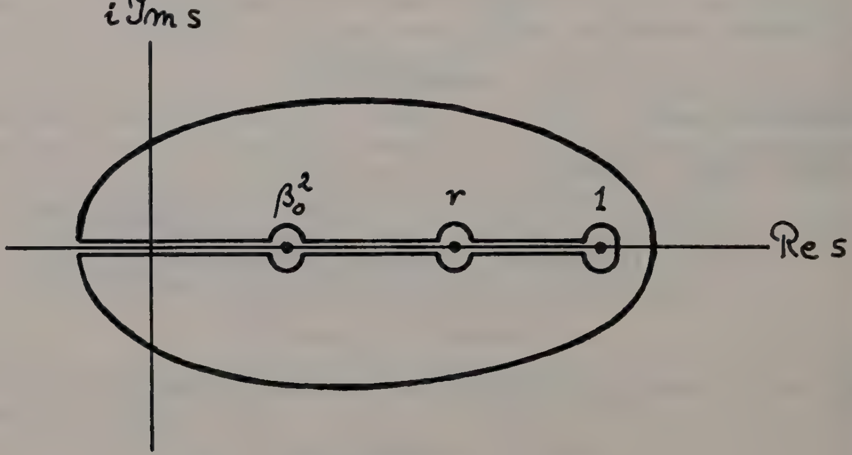


Fig. 10. Path of integration in the s-plane.

This integral occurs in Eq. (19). It can be split up into the two integrals

$$I_1=\int_{\beta_0^2}^{1}\frac{\sqrt{(1-s)\,(s-\beta_0^2)}\cdot(s-2k^2)^2}{(s-r)\,[(s-r_0)^2+r_1^2]}\,ds,$$

$$I_2=4k^3\int_{\beta_0^2}^{k^2}\frac{\sqrt{(k^2-s)\,(s-\beta_0^2)}\cdot(1-s)}{(s-r)\,[(s-r_0)^2+r_1^2]}\,ds.$$

r_0 and r_1 are defined in Appendix 1.

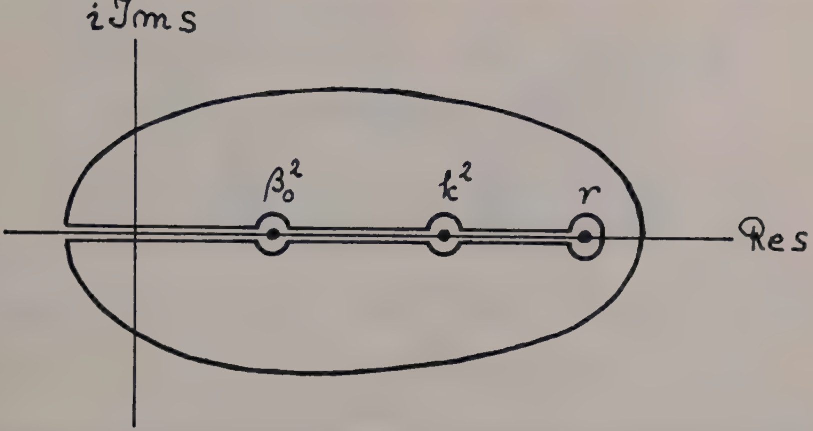


Fig. 11. Path of integration in the s-plane.

By choosing a path of integration shown in Fig. 10 for I_1 and in Fig. 11 for I_2, the two integrals are easily calculated. It is to be observed that the poles $s = r_0 \pm i r_1$ can be disregarded, because the corresponding residues cancel when adding I_1 and I_2.

One obtains

$$I_1 + I_2 = \begin{cases} (M - r) & \text{for} \quad \beta_0^2 < r < k^2 \\ (M - r) + \pi \cdot 4\,k^3\,\dfrac{r\,(r - k_s^2)\,(1 - r)\,\sqrt{(r - k^2)\,(r - \beta_0^2)}}{(r - 2\,k^2)^4 + 16\,k^6\,(1 - r)\,(r - k^2)} & \text{for} \quad k^2 < r < 1\,, \end{cases}$$

where M is a constant.

APPENDIX 3

Demonstration that $\varphi(r) = Q_1 \cdot \dfrac{(r - 2\,k^2)^2}{(r - k_s^2)\,\sqrt{(1 - r)\,(r - \beta_0^2)^3}}$ *is not a solution to Problem C.*

This demonstration is completely analogous to the calculation of u_z in Problem D. One obtains

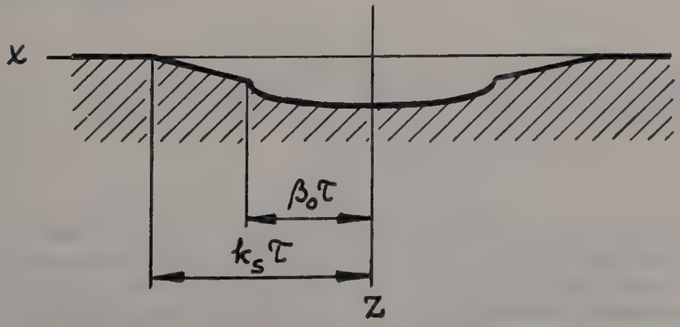


Fig. 12. Surface position at a fixed time, corresponding to the false solution when $\beta_0 < k_s$.

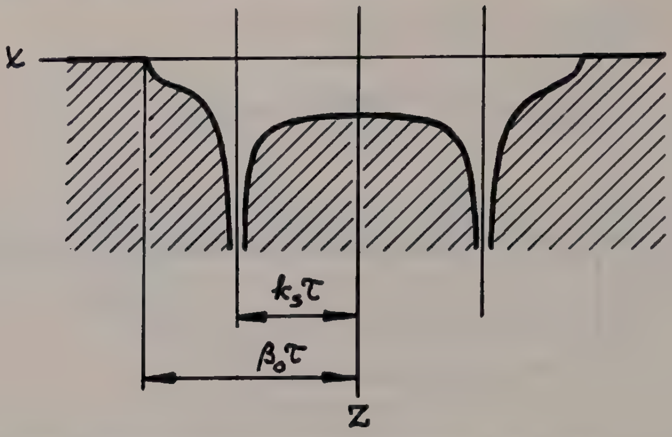


Fig. 13. Surface position at a fixed time, corresponding to the false solution when $\beta_0 > k_s$.

$$\frac{\partial^2 u_z}{\partial \tau^2} = \begin{cases} \dfrac{\pi Q_1}{2\varrho c_d^2 k_s} \cdot \dfrac{\delta\left(\tau - \dfrac{|x|}{k_s}\right)}{\sqrt{\left(1-\dfrac{\beta_0^2}{k_s^2}\right)^3}} - \dfrac{Q_1}{\varrho c_d^2 |x|} \cdot \dfrac{U\left(\tau - \dfrac{|x|}{\beta_0}\right)}{\left(1-\dfrac{k_s^2\tau^2}{x^2}\right)\sqrt{\left(\beta_0^2\dfrac{\tau^2}{x^2}-1\right)^3}} & \text{for} \quad \beta_0 < k_s \\[2ex] -\dfrac{Q_1}{\varrho c_d^2 |x|} \cdot \dfrac{U\left(\tau - \dfrac{|x|}{\beta_0}\right)}{\left(1-k_s^2\dfrac{\tau^2}{x^2}\right)\sqrt{\left(\beta_0^2\dfrac{\tau^2}{x^2}-1\right)^3}} & \text{for} \quad \beta_0 > k_s. \end{cases}$$

The solution for $\beta_0 < k_s$ clearly does not satisfy Problem C. As for the solution for $\beta_0 > k_s$, after two integrations one obtains

$$u_z = \frac{Q_1\beta_0}{\beta_0^2-k_s^2}\sqrt{\tau^2-\frac{x^2}{\beta_0^2}} + \frac{Q_1 k_s^2}{\sqrt{(\beta_0^2-k_s^2)^3}} \cdot \left\{\left(\tau-\frac{|x|}{k_s}\right)\cdot \ln\left|\frac{\tau-\dfrac{k_s|x|}{\beta_0^2}+\sqrt{1-\dfrac{k_s^2}{\beta_0^2}}\sqrt{\tau^2-\dfrac{x^2}{\beta_0^2}}}{\sqrt{1-\dfrac{k_s^2}{\beta_0^2}}\left(\tau-\dfrac{|x|}{k_s}\right)}\right| - \right.$$

$$\left. -\left(\tau+\frac{|x|}{k_s}\right)\ln\left|\frac{\tau+\dfrac{k_s|x|}{\beta_0^2}-\sqrt{1-\dfrac{k_s^2}{\beta_0^2}}\sqrt{\tau^2-\dfrac{x^2}{\beta_0^2}}}{\sqrt{1-\dfrac{k_s^2}{\beta_0^2}}\left(\tau+\dfrac{|x|}{k_s}\right)}\right|\right\} + \frac{Q_1 k_s|x|}{\sqrt{(\beta_0^2-k_s^2)^3}} \cdot \ln\frac{k_s^2}{\beta_0^2-k_s^2}.$$

In Figs. 12 and 13 the surface position at a fixed time is sketched.

It is seen that the solution for $\beta_0 < k_s$ does not satisfy the requirement $u_z = 0$ for $|x| \geqslant \beta_0\tau$ (although $\partial^2 u_z/\partial\tau^2 = 0$ for $|x| > \beta_0\tau$ with the exception $|x| = k_s\tau$) and that the solution for $\beta_0 > k_s$ does not satisfy the requirement that u_z should be finite (although $u_z = 0$ for $|x| > \beta_0\tau$).

The occurrence of the false solution to Problem C seems to be a result of the change of boundary condition from

$$u_z = 0 \text{ for } |x| \geqslant \beta_0 \tau \text{ to } \frac{\partial^2 u_z}{\partial \tau^2} = 0 \text{ for } |x| > \beta_0 \tau.$$

APPENDIX 4

Calculation of Q

The quantity Q occurring in Eqs. (23) and (24) is to be determined from the equation

$$Q - \frac{1}{\pi} \int\limits_{\beta_0^2}^{1} \frac{\sqrt{s} - \beta_0}{\sqrt{s}} \varphi(s)\, ds = \frac{2 q_0 \beta_0}{\pi}.$$

Thus
$$Q = \frac{2 q_0 \beta_0}{\pi - \int\limits_{\beta_0^2}^{1} \frac{\sqrt{s} - \beta_0}{\sqrt{s}} \cdot \frac{\varphi(s)}{Q}\, ds},$$

where $\varphi(s)/Q$ is given by Eqs. (23) and (24). One obtains

$$\int\limits_{\beta_0^2}^{1} \frac{\sqrt{s} - \beta_0}{\sqrt{s}} \cdot \frac{\varphi(s)}{Q}\, ds = \int\limits_{\beta_0^2}^{k^2} \frac{\sqrt{s} - \beta_0}{\sqrt{s}} \cdot \frac{(s - 2k^2)^2 - 4k^3 \sqrt{(1-s)(k^2 - s)}}{s\sqrt{(1-s)(s - \beta_0^2)^3}}\, ds +$$

$$+ \int\limits_{k^2}^{1} \frac{\sqrt{s} - \beta_0}{\sqrt{s}} \cdot \frac{(s - 2k^2)^2}{s\sqrt{(1-s)(s - \beta_0^2)^3}}\, ds =$$

$$= \int\limits_{\beta_0^2}^{1} \frac{(\sqrt{s} - \beta_0)(s - 2k^2)^2}{s\sqrt{s(1-s)(s - \beta_0^2)^3}}\, ds - 4k^3 \int\limits_{\beta_0^2}^{k^2} \frac{(\sqrt{s} - \beta_0)\sqrt{k^2 - s}}{s\sqrt{s(s - \beta_0^2)^3}}\, ds =$$

$$= \lim_{r \to \beta_0^2} \left\{ \int\limits_{r}^{1} \frac{(s - 2k^2)^2}{s\sqrt{(1-s)(s - \beta_0^2)^3}}\, ds - \int\limits_{r}^{1} \frac{\beta_0^2 (s - 2k^2)^2}{s\sqrt{s(1-s)(s - \beta_0^2)^3}}\, ds - \right.$$

$$\left. - 4k^3 \int\limits_{r}^{k^2} \frac{\sqrt{k^2 - s}}{s\sqrt{(s - \beta_0^2)^3}}\, ds + 4k^3 \int\limits_{r}^{k^2} \frac{\beta_0^2 \sqrt{k^2 - s}}{s\sqrt{s(s - \beta_0^2)^3}}\, ds \right\}.$$

The second and the fourth integral in the last member are calculated in Appendix 5. The first integral, I_1, and the third integral, I_3, are easily calculated by

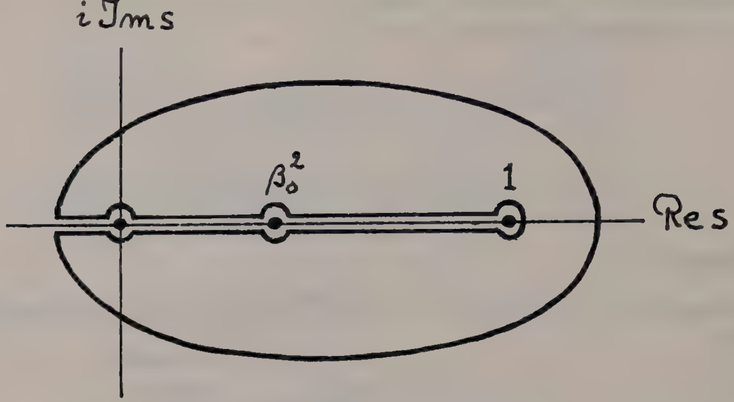


Fig. 14. Path of integration in the s-plane.

contour integration, choosing a path of integration for I_1 as shown in Fig. 14, and for I_3 as shown in Fig. 15. The radius of the semi-circles at $s=\beta_0^2$ is $(r-\beta_0^2)$. When $r \to \beta_0^2$ one obtains

$$I_1 = \int_r^1 \frac{(s-2\,k^2)^2}{s\sqrt{(1-s)\,(s-\beta_0^2)^3}}\,d\,s \to \pi - \pi \cdot \frac{4\,k^4}{\beta_0^3} + \frac{2\,(\beta_0^2-2\,k^2)^2}{\beta_0^2\sqrt{(1-\beta_0^2)\,(r-\beta_0^2)}}$$

$$I_3 = \int_r^{k^2} \frac{\sqrt{k^2-s}}{s\sqrt{(s-\beta_0^2)^3}}\,d\,s \to -\pi \cdot \frac{k}{\beta_0^3} + \frac{2\sqrt{k^2-\beta_0^2}}{\beta_0^2\sqrt{r-\beta_0^2}}.$$

Thus we get

$$\int_{\beta_0^2}^1 \frac{\sqrt{s}-\beta_0}{\sqrt{s}} \cdot \frac{\varphi\,(s)}{Q}\,d\,s = \pi - 2 \cdot \frac{\beta_0^2-4\,k^2\,\beta_0^2+4\,k^4}{\beta_0\,(1-\beta_0^2)}\,\mathbf{K}\,(\sqrt{1-\beta_0^2}) +$$

$$+ 2 \cdot \frac{\beta_0^4-4\,k^2\,(1+k^2)\,\beta_0^2+8\,k^4}{\beta_0^3\,(1-\beta_0^2)}\,\mathbf{E}\,(\sqrt{1-\beta_0^2}) +$$

$$+ \frac{8\,k^2}{\beta_0}\,\mathbf{K}\left(\sqrt{1-\left(\frac{\beta_0}{k}\right)^2}\right) - \frac{16\,k^4}{\beta_0^3}\,\mathbf{E}\left(\sqrt{1-\left(\frac{\beta_0}{k}\right)^2}\right)$$

and then
$$Q = q_0 \cdot \frac{\beta_0^2\,(1-\beta_0^2)}{g\,(\beta_0)},$$

where
$$g\,(\beta_0) = [(1-4\,k^2)\,\beta_0^2+4\,k^4]\,\mathbf{K}\,(\sqrt{1-\beta_0^2}) -$$

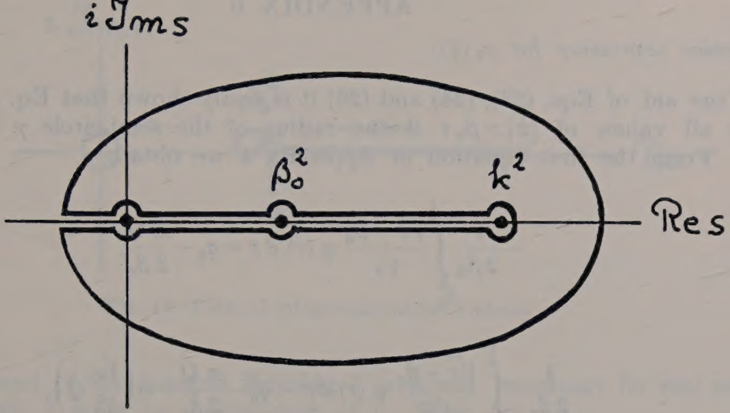


Fig. 15. Path of integration in the s-plane.

$$-\frac{1}{\beta_0^2}\left[\beta_0^4-4\,k^2\,(1+k^2)\,\beta_0^2+8\,k^4\right]\mathrm{E}\left(\sqrt{1-\beta_0^2}\right)-$$

$$-4\,k^2\,(1-\beta_0^2)\,\mathrm{K}\left(\sqrt{1-\left(\frac{\beta_0}{k}\right)^2}\right)+\frac{8\,k^4}{\beta_0^2}\,(1-\beta_0^2)\,\mathrm{E}\left(\sqrt{1-\left(\frac{\beta_0}{k}\right)^2}\right).$$

APPENDIX 5

Calculation of $\sigma_z(\xi)$

From Eqs. (23), (24) and (26) we get

$$-\frac{2\,\sigma_z(\xi)}{Q}=\int\limits_{\xi^2}^{1}\frac{(r-2\,k^2)^2}{r\sqrt{r\,(1-r)\,(r-\beta_0^2)^3}}\,d\,r-4\,k^3\int\limits_{\xi^2}^{k^2}\frac{\sqrt{k^2-r}}{r\sqrt{r\,(r-\beta_0^2)^3}}\,d\,r,$$

where $\xi^2=x^2/\tau^2$. If $\xi>k$ the last term should be dropped. Inserting $\beta^2=r$, we obtain after some rearranging:

$$-\frac{2\,\sigma_z(\xi)}{Q}=2\int\limits_{\xi}^{1}\frac{d\,\beta}{\sqrt{(1-\beta^2)\,(\beta^2-\beta_0^2)}}+\frac{2}{\beta_0^2}\,(\beta_0^2-2\,k^2)^2\int\limits_{\xi}^{1}\frac{d\,\beta}{\sqrt{(1-\beta^2)\,(\beta^2-\beta_0^2)^3}}-$$

$$-\frac{8\,k^4}{\beta_0^2}\int\limits_{\xi}^{1}\frac{d\,\beta}{\beta^2\sqrt{(1-\beta^2)\,(\beta^2-\beta_0^2)}}-\frac{8\,k^3}{\beta_0^2}\int\limits_{\xi}^{k}\frac{\sqrt{k^2-\beta^2}}{\sqrt{(\beta^2-\beta_0^2)^3}}\,d\,\beta+\frac{8\,k^3}{\beta_0^2}\int\limits_{\xi}^{k}\frac{\sqrt{k^2-\beta^2}}{\beta^2\sqrt{\beta^2-\beta_0^2}}\,d\,\beta.$$

These integrals can all be reduced to elementary integrals and elliptic integrals of the first and the second kind. The result is given by Eqs. (27) and (28).

APPENDIX 6

Alternative expression for $\sigma_z(\xi)$

By the aid of Eqs. (23), (24) and (26) it is easily shown that Eq. (31) expresses σ_z for all values of $|x| > \beta_0 \tau$ if the radius of the semi-circle γ is sufficiently small. From the first equation in Appendix 4 we obtain

$$-\frac{1}{2\beta_0}\int_{\beta_0^2}^{1}\frac{\sqrt{r}-\beta_0}{\sqrt{r}}\,\varphi(r)\,dr = q_0 - \frac{\pi Q}{2\beta_0}.$$

Thus
$$-\frac{1}{2\beta_0}\int_{\beta_0^2+\varepsilon}^{1}\frac{\sqrt{r}-\beta_0}{\sqrt{r}}\,\varphi(r)\,dr = q_0 - \frac{\pi Q}{2\beta_0} + 0\left(\frac{\sqrt{\varepsilon}}{\beta_0^2}Q\right),$$

$$\frac{1}{2}\int_{\beta_0^2+\varepsilon}^{1}\frac{\varphi(r)}{\sqrt{r}}\,dr = q_0 - \frac{\pi Q}{2\beta_0} + \frac{1}{2\beta_0}\int_{\beta_0^2+\varepsilon}^{1}\varphi(r)\,dr + 0\left(\frac{\sqrt{\varepsilon}}{\beta_0^2}Q\right)$$

when $\varepsilon \ll 1$. The integral in the right member has been calculated in Appendix 4. We obtain

$$\frac{1}{2}\int_{\beta_0^2+\varepsilon}^{1}\frac{\varphi(r)}{\sqrt{r}}\,dr = q_0 + \frac{Q\left[(\beta_0^2-2k^2)^2-4k^3\sqrt{(1-\beta_0^2)(k^2-\beta_0^2)}\right]}{\beta_0^3\sqrt{1-\beta_0^2}\sqrt{\varepsilon}} + 0\left(\frac{\sqrt{\varepsilon}}{\beta_0^2}Q\right).$$

Now we study the integral

$$I = -\tfrac{1}{2}\,Re\,Q\int_{\Gamma(r)}\frac{(r-2k^2)^2-4k^3\sqrt{(1-r)(k^2-r)}}{r\sqrt{r(1-r)(r-\beta_0^2)^3}}\,dr,$$

where $\Gamma(r)$ is the path of integration shown in Fig. 16.

Then obviously

$$I = -\tfrac{1}{2}\,Re\,Q\int_{0}^{\pi}\frac{(\beta_0^2-2k^2)^2-4k^3\sqrt{(1-\beta_0^2)(k^2-\beta_0^2)}}{\beta_0^3\sqrt{1-\beta_0^2}\sqrt{\varepsilon}}\cdot i\,e^{-i\frac{\varphi}{2}}\,d\varphi -$$

$$-\frac{1}{2}\int_{\beta_0^2+\varepsilon}^{1}\frac{\varphi(r)}{\sqrt{r}}\,dr + 0\left(\frac{\sqrt{\varepsilon}}{\beta_0^2}Q\right) =$$

$$= \frac{Q\left[(\beta_0^2-2k^2)^2-4k^3\sqrt{(1-\beta_0^2)(k^2-\beta_0^2)}\right]}{\beta_0^3\sqrt{1-\beta_0^2}\sqrt{\varepsilon}} - \frac{1}{2}\int_{\beta_0^2+\varepsilon}^{1}\frac{\varphi(r)}{\sqrt{r}}\,dr + 0\left(\frac{\sqrt{\varepsilon}}{\beta_0^2}Q\right) = -q_0 + 0\left(\frac{\sqrt{\varepsilon}}{\beta_0^2}Q\right).$$

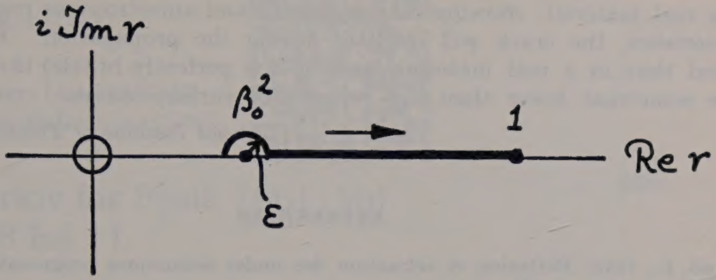


Fig. 16. Path of integration in the r-plane.

As the integrand of the integral defining I is purely imaginary for real positive values of $r < \beta_0^2$, I must be independent of ε. Thus $I = -q_0$.

After inserting $r = x^2/\tau^2$, one finds that

$$I = -\frac{4\,k^4\,Q}{x}\,Re \int\limits_{\Gamma_1+\gamma} \frac{\left(\dfrac{1}{2\,k^2} - \dfrac{\tau^2}{x^2}\right)^2 - \dfrac{\tau^2}{x^2}\sqrt{\left(\dfrac{\tau^2}{x^2} - 1\right)\left(\dfrac{\tau^2}{x^2} - \dfrac{1}{k^2}\right)}}{\sqrt{\left(\dfrac{\tau^2}{x^2} - 1\right)\left(1 - \beta_0^2\,\dfrac{\tau^2}{x^2}\right)^3}}\,d\tau,$$

where Γ_1 and γ are shown in Fig. 5.

But $I = -q_0$. Further

$$Re \int\limits_{\Gamma_2(\tau)} \frac{\left(\dfrac{1}{2\,k^2} - \dfrac{\tau^2}{x^2}\right)^2 - \dfrac{\tau^2}{x^2}\sqrt{\left(\dfrac{\tau^2}{x^2} - 1\right)\left(\dfrac{\tau^2}{x^2} - \dfrac{1}{k^2}\right)}}{\sqrt{\left(\dfrac{\tau^2}{x^2} - 1\right)\left(1 - \beta_0^2\,\dfrac{\tau^2}{x^2}\right)^3}}\,d\tau = 0,$$

where $\Gamma_2(\tau)$ is the path of integration shown in Fig. 5. This follows from the fact that the integrand is purely imaginary for $\tau > |x|/\beta_0$. Thus, if $\tau > |x|/\beta_0$, the right member of Eq. (31) equals $I = -q_0$, and Eq. (31) thus expresses σ_z for all values of $|x| \neq \beta_0\,\tau$ if γ is sufficiently small.

Conclusions

The investigations have shown that a two-dimensional crack will propagate in one plane in a linear elastic medium which is homogeneous and isotropic as regards stress-strain relations and fracturing characteristics. This conclusion is partly based on simple specific assumptions regarding the fracture criterion for such a material. The shape of the crack is elliptic, just as in the static case, although the eccentricity is somewhat larger than in the static case. The velocity of the crack-tip equals the velocity of surface (Rayleigh) waves.

In a real material, showing inhomogeneity and anisotropy as regards fracturing characteristics, the crack will oscillate during the propagation. Thus it may be expected that in a real material (even if it is perfectly brittle) the crack velocity will be somewhat lower than the velocity of surface waves.

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Tryckt den 4 juli 1960

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